ON SOME NUMERICAL SERIES RELATED TO SPECIAL PSEUDO-ORTHOGONAL GROUPS

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Let $e_{p,q}$ be a diagonal matrix $\mathrm{diag}\{1,\dots,1,-1,\dots,-1\}$ of order $p+q$ with p of 1 and q of -1. The special pseudo-orthogonal group $\mathrm{SO}(p,q)$ consists of unimodular real matrices g of order $p+q$ satisfying the equation $e_{p,q}g^{\mathrm{t}} = g^{-1}e_{p,q}$. We assume that $p \geqslant q$. Let X_0 denote the cone

$$\left\{x=(x_1,\,\dots,x_{p+q}) : \sum_{i=1}^{p} x_i^2 - \sum_{i=1}^{q} x_{p+i}^2 = 0, x \neq 0\right\}$$

in $\mathbb{R}^{p+q}$. It is a homogeneous space of the group $\mathrm{SO}(p,q)$. Let us consider the following subsets of the cone X_0:

$$\begin{aligned}
\Gamma_1 &:= \left\{x : \sum_{i=1}^{p+q} x_i^2 = 1\right\},\\
\Gamma_2 &:= \{x : x_p + x_{p+q} = 1\},\\
\Gamma_3 &:= \{x : x_p = \pm 1\},\\
\Gamma_4(n) &:= \left\{x : \sum_{i=1}^{n} x_i^2 = 1\right\},\\
\Gamma_5(n,m) &:= \left\{x : \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{m} x_{p+i}^2 = 1\right\},
\end{aligned}$$

where $n \in \{2,\,\dots,\,p-1\}$, $m \in \{2,\,\dots,\,q-1\}$. For $i=1,\dots,5$, there exists a subgroup G_i of $\mathrm{SO}(p,q)$ which acts on Γ_i transitively. Namely, $G_1 \simeq \mathrm{SO}(p)\times\mathrm{SO}(q)$, $G_3 \simeq \mathrm{SO}(p-1)\times\mathrm{SO}(q)$, $G_4 \simeq \mathrm{SO}(n)\times\mathrm{SO}(p-n,q)$, $G_5 \simeq \mathrm{SO}(n,m)\times\mathrm{SO}(p-n,q-m)$ and $G_2 \equiv G_2(p,q)$ is a nilpotent subgroup given by formulas

$$\begin{aligned}
G_2(2,1) &= \exp[\mathbb{R}(e_{12}+e_{13}+e_{31}-e_{21})],\\
G_2(2,2) &= \exp[\mathbb{R}(e_{12}-e_{21}-e_{23}-e_{32})+\mathbb{R}(e_{14}+e_{41}+e_{43}-e_{34})],\\
G_2(3,2) &= \exp[\mathbb{R}(e_{12}-e_{21}-e_{23}-e_{32})+\\
&+ \mathbb{R}(e_{15}+e_{51}+e_{53}-e_{35})+\mathbb{R}(e_{14}+e_{41}+e_{43}-e_{34})],
\end{aligned}$$

etc., where e_{ij} are vectors of the canonical basis in $\mathrm{Mat}(p+q,\mathbb{R})$. Let $d\mu_i(x)$ denote a G_i-invariant measure on Γ_i.

For $\sigma \in \mathbb{C}$, let D_σ denote the linear space of σ-homogeneous functions f on X_0 of class C^∞. The representation T_σ of $\mathrm{SO}(p,q)$ acts by translations on D_σ:

$$T_\sigma(g)f(x) = f(g^{-1}x).$$

We define bilinear functionals F_i, $i = 1, \dots, 5$, on the pair $D_\sigma, D_{\widehat{\sigma}}$ as follows:

$$F_i(f_1, f_2) = \int_{\Gamma_i} f_1(x) f_2(x)\, d\mu_i(x).$$

Proposition 1 *If $\widehat{\sigma} = -\sigma - p - q + 2$, then all 5 functionals F_i, $i = 1, \dots, 5$, coincide.*

Indeed, we can write a point x of the cone X_0 as $x = ty$, where $y \in \Gamma_i$ and $t \in \mathbb{R}$. Then $t^{p+q-3} d\mu_i(y)$ is a measure on X_0 invariant with respect to $\mathrm{SO}(p,q)$. It remains to remember that functions in D_σ are σ-homogeneous.

We want to establish some relations for special functions using the formula

$$F_i(u, v) = F_j(u, v), \quad i \neq j,$$

where u and v are some functions.

Let $\{f_K^{\sigma i}\}$ be a basis of D_σ consisting of eigenfunctions for the subgroup G_i. Suppose that an automorphism $c^{\sigma ij}$ transforms $\{f_K^{\sigma i}\}$ into $\left\{f_M^{\widehat{\sigma} j}\right\}$. Then

$$c_{MK}^{\sigma ij} = F_i(f_K^{\sigma i}, f_{-M}^{\widehat{\sigma} j})$$

is a matrix element of $c^{\sigma ij}$. If $i \neq j$, we sometimes can obtain interest relations between special functions by the formula

$$F_i(f_K^{\sigma i}, f_{-M}^{\widehat{\sigma} j}) = F_j(f_K^{\sigma i}, f_{-M}^{\widehat{\sigma} j}).$$

Now define functionals $\widetilde{F}_i(g)$:

$$\widetilde{F}_i(g)(f_1, f_2) := F_i(T_\sigma(g) f_1,\, T_{\widehat{\sigma}}(g) f_2).$$

The formula

$$t_{K\widetilde{K}}^{\sigma i}(g) = F_i(T_\sigma(g) f_K^{\sigma i},\, f_{-\widetilde{K}}^{\widehat{\sigma} i})$$

gives matrix elements of the representation T_σ associated with the basis $\{f_K^{\sigma i}\}$. If $i \neq j$, we can obtain some new relations by

$$F_i(T_\sigma(g) f_K^{\sigma i},\, f_{-\widetilde{K}}^{\widehat{\sigma} i}) = F_j(T_\sigma(g) f_K^{\sigma i},\, f_{-\widetilde{K}}^{\widehat{\sigma} i}).$$

The following proposition can be useful to compute $t_{K\widetilde{K}}^{\sigma i}(g)$.

Proposition 2 *If $\widehat{\sigma} = -\sigma - p - q + 2$, then $\widetilde{F}_i(g) = F_i$.*

The proof consists of three steps.

Step 1. Since $\widehat{\sigma} = -\sigma - p - q + 2$, then (according to Proposition 1) $\widetilde{F}_i = \widetilde{F}_1$.

Let us use the decomposition

$$\mathrm{SO}(p,q) = G_1 \times H \times G_1,$$

where H is the subgroup consisting of elements h having the form:

$$h = h(t_1, \ldots, t_q) = \prod_{i=1}^{q} h_{(p-q+i, p+q-i+1)}(t_i),$$

where $h_{(k,l)}$ sends x to x' such that

$$\begin{aligned} x'_k &= x_k \cosh t + x_l \sinh t, \\ x'_l &= x_k \sinh t + x_l \cosh t, \\ x'_j &= x_j, \ j \notin \{k, l\}. \end{aligned}$$

Step 2. For any $g \in G_1$ and any $\sigma, \widehat{\sigma} \in \mathbb{C}$, we have $\widetilde{F}_i(g) = F_i$.

Step 3. If $g \in H$ and $\widehat{\sigma} = -\sigma - p - q + 2$, then $\widetilde{F}_i(g) = F_i$.

For example, let us consider the group $\mathrm{SO}(2,2)$ and bases

$$\left\{ f^{\sigma 1}_{(k_1, k_2)} : \ k_1 \in \mathbb{N} \cup \{0\}, \ k_2 \in \{-k_1, -k_1 + 1, \ldots, k_1\} \right\}$$

and

$$\left\{ f^{\sigma 2}_{(l_1, l_2)} : \ (l_1, l_2) \in \mathbb{R}^2 \right\},$$

consisting of functions

$$f^{\sigma 1}_{(k_1, k_2)} = \left(x_1^2 + x_2^2\right)^{\frac{\sigma - k_1 - k_2}{2}} (x_2 + i x_1)^{k_1} (x_4 + i x_3)^{k_2}$$

and

$$f^{\sigma 2}_{(l_1, l_2)} = (x_2 + x_4)^{\sigma} \exp\left(i \, \frac{l_1 x_1 - l_2 x_3}{x_2 + x_4} \right).$$

Let

$$u(x) = (x_1 \cosh\beta - x_3 \sinh\beta)^{\sigma}.$$

The formula

$$F_1(u, f^{-\sigma-2,2}_{(l_1, l_2)}) = F_2(u, f^{-\sigma-2,2}_{(l_1, l_2)})$$

implies [1] the following identities

$$\begin{aligned} &\sum_{k_1=0}^{\infty} \sum_{k_2=-k_1}^{k_1} (k_1^2 + k_2^2)^{1/4} \left[\Gamma\left(\frac{k_1 + k_2}{2} + \frac{3}{4}\right) \Gamma\left(\frac{k_2 - k_1}{2} + \frac{3}{4}\right) \right]^{-1} \times \\ &\times W_{\frac{k_1 + k_2}{2}, -\frac{1}{4}}(l_2 - l_1) W_{\frac{k_2 - k_1}{2}, -\frac{1}{4}}(l_2 + l_1) K_{-\frac{1}{2}}\left(\exp(-\beta) \left(k_1^2 + k_2^2\right)^{1/2}\right) \\ &= \frac{4}{3} \pi^{-1} l_2^{-1/2} [\sinh\beta]^{-1/2} [\cosh\beta]^{-1} [\sin(-1/2)]^{-1} \times \\ &\times \exp\beta \ \left(l_2^2 - l_1^2\right)^{1/4} \left(1 + \left(-l_1 l_2^{-1} \tanh\beta\right)^{1/2}\right)^{-1}, \end{aligned}$$

and

$$\sum_{k_1=0}^{\infty}\sum_{k_2=-k_1}^{k_1}\left[\Gamma\left(\frac{k_1+k_2}{2}+\frac{3}{4}\right)\right]^{-1}\times$$
$$\times\quad\left[\Gamma\left(\frac{k_2-k_1}{2}+\frac{3}{4}\right)\right]^{-1}W_{\frac{k_1+k_2}{2},0}(l_2-l_1)\times$$
$$\times\quad W_{\frac{k_2-k_1}{2},0}(l_2+l_1)K_0\left(\exp(-\beta)\left(k_1^2+k_2^2\right)^{1/2}\right)=0,$$

which are true under the condition $\beta>0$, $l_2>0$, $|l_1|<l_2$.

If we take the simplest Lorentz group $SO(2,1)$, then bases $\{f_k^{\sigma 1}\,:\,k\in\mathbb{Z}\}$ and $\{f_l^{\sigma 2}\,:\,l\in\mathbb{R}\}$ consist of functions

$$f_k^{\sigma 1}=\left(x_1^2+x_2^2\right)^{\frac{\sigma-k}{2}}(x_2+ix_1)^k$$

and

$$f_l^{\sigma 2}=(x_2+x_3)^{\sigma}\exp\left(\frac{ilx_1}{x_2+x_3}\right).$$

From the formula

$$c_{kl}^{\sigma 12}=F_1\left(f_k^{\sigma 1},f_{-l}^{-\sigma-1,2}\right)=F_2\left(f_k^{\sigma 1},f_{-l}^{-\sigma-1,2}\right)$$

we derive [2] a representation of Whittaker function:

$$W_{k\mathrm{sgn}l,\sigma+1/2}\,(2|l|)=|l|^{\sigma+1/2}\Gamma(k\mathrm{sgn}l-\sigma)\times$$
$$\times\quad\sum_{n=0}^{\infty}\sum_{m=0}^{k}l^n\binom{k}{m}\Big[\left(1+3^{3n-m}\right)\mathrm{B}\left(\frac{n+m+1}{2},k-m+1\right){}_2F_1\left(\begin{matrix}\frac{n-m+1}{2}-\sigma,k-m+1;\\ \frac{n-m+3}{2}+k;-1\end{matrix}\right)+$$
$$+\quad\left(1+i^{2k+3n+m}\right)\mathrm{B}\left(\frac{m-n+1}{2}+\sigma,k-m+1\right){}_2F_1\left(\begin{matrix}\frac{-n-m+1}{2}-\sigma,k-m+1;\\ \frac{-n-m+3}{2}+\sigma+k;-1\end{matrix}\right)\Big],$$

which holds under the condition $k\in\mathbb{N}$, $-1<\mathrm{Re}\,\sigma<0$.

In [2] we obtained the matrix elements $t_{k\tilde{k}}^{-\sigma-1,1}(g)$ where g belongs to the subgroup $G_2\times H$ of the group $\mathrm{SO}(2,1)$. This result implies immediately the following equality

$$A(s,t)+A(-s,-t)=\delta_{st}$$

for the special function

$$A(s,t):=\frac{1}{2}(t-\sigma)\Gamma(\sigma+s+1)\times$$
$$\times\quad\Big[(-1)^{\sigma}\frac{\Gamma(2\sigma+1)\Gamma(2\sigma+3)\Gamma(\sigma+s+3)}{\Gamma(-\sigma-t)}\;{}_3F_2(\sigma+s+1,2\sigma+3,2;2\sigma+2,-\frac{5}{2};1)+$$
$$+\quad\frac{\Gamma(2\sigma+1)\Gamma(\sigma-t+1)\Gamma(\frac{3}{2}-\sigma-t)}{\Gamma(\sigma+t+1)}\;{}_3F_2(-\sigma-s,-2\sigma+1,2;-2\sigma,-\sigma+k-2;1)+$$
$$+\quad(-1)^{\sigma+1}\frac{\Gamma(-2\sigma-1)\Gamma(2\sigma+3)}{\Gamma(s-\sigma)\Gamma(\sigma-t+3)}\;{}_3F_2(\sigma+s+1,2\sigma+3,2;2\sigma+2,\sigma-t+3;1)\Big].$$

References

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[2] I. A. Shilin. Formulas for special functions of mathematical physics connected with unimodular pseudo-orthogonal groups. Matem. modelir., 2004, tom 16, No. 12, 11–19.